LOCAL CONVEXITY AND STARSHAPED SETS

BY

F. A. VALENTINE

ABSTRACT

Previously [7] we proved among other results that a closed connected set in E_n which has a unique point of local nonconvexity is starshaped. Here we characterize a fairly large class of plane sets whose points of local nonconvexity are so arranged that starshapedness follows. This theory determines as a special case the simple closed polygonal regions which are starshaped. In order to proceed simply we utilize the following notations and definitions.

NOTATIONS. The interior, closure and boundary of a set S in Euclidean n-space E_n are denoted by $\operatorname{int} S$, $\operatorname{cl} S$ and $\operatorname{bd} S$ respectively. The closed segment joining points $x \in E_n$, $y \in E_n$ is denoted by xy. Set union, intersection and difference are indicated by $\cup, \cap$ and $\sim$ respectively. The symbol conv S denotes the convex hull of the set S. The symbol 0 denotes the empty set.

DEFINITION 1. A point $x \in S$ is a point of local convexity of S if there exists a neighborhood N of x such that $N \cap S$ is convex; otherwise x is called a point of local nonconvexity. The set of all points of local nonconvexity of S is denoted by Q.

DEFINITION 2. A set S is starshaped with respect to a point p if $px \subset S$ for all points $x \in S$.

DEFINITION 3. The two closed rays of a line L which have only a point $x \in L$ in common are called complementary rays. If $R(x)$ is a ray with endpoint x, its complementary ray is denoted by $R'(x)$.

DEFINITION 4. A ray $R(x)$ with endpoint $x \in bd\, S$ is an external ray of support to the int S if $R(x) \cap \operatorname{int} S = 0$.

DEFINITION 5. If x is a boundary point of a set S, then $K(x)$ is the union of all the external rays of support to int S at x. A boundary point $x \in S$ is called a one-sided point of external support of int S if

$$K(x) \neq 0,$$

and if $K(x)$ lies in a closed half-space which contains x in its boundary. The set $K(x)$ is called an external cone of support. The union of all the complementary rays $R'(x)$ where $R(x) \subset K(x)$ is denoted by

$$K'(x).$$

Received May 4, 1965

The preparation of this paper was supported in part by NSF Grant GP-1988.

The following theorem is one of Krasnoselskii type [2], and it is also related to results developed by Eugene Robkin in his thesis [5]. For another kind of theorem of Krasnoselskii type for polygonal regions see Molnar [4].

THEOREM 1. *Suppose $S \subset E_2$ is a bounded plane set which is the closure of an open connected set.*

Then S is starshaped if and only if both of the following conditions hold.

(a) *Each point of local nonconvexity $x \in S$ has a nonempty cone of external support $K(x)$ to* int$S$ *at x.*

(b) *If x_1, x_2, x_3 are three points of local nonconvexity of S (they need not be distinct) which are also one-sided points of external support to* int S, *then there exist three external rays of support $R(x_1), R(x_2), R(x_3)$ to* int S *at x_1, x_2, x_3 respectively whose corresponding complementary rays $R'(x_1), R'(x_2), R'(x_3)$ are concurrent and meet in S, so that*

$$S \cap R'(x_1) \cap R'(x_2) \cap R'(x_3) \neq 0. \tag{1}$$

Proof. We will prove that conditions (a) and (b) imply that S is starshaped. Let Q_1 denote the set of all points of local nonconvexity of S which are also one-sided points of external support to int S (see Definitions 1 and 5). First, observe that if S is convex then $Q_1 = 0$; however, in this case S is also starshaped, being convex. Hence, suppose S is not convex. Since int $S \neq 0$, when int S is not convex a theorem of Leja and Wilkosz [3] implies that a point q of local nonconvexity of S exists which is the midpoint of the bounding diameter of a closed semicircular region which, except for q, lies in int S. Clearly such a point q is in Q_1, so that $Q_1 \neq 0$.

Now define $C(x)$ as follows,

$$C(x) \equiv (\text{conv } K'(x)) \cap \text{conv } S$$

where $x \in Q_1$ (see Definition 5). Since conv $K'(x)$ is closed, and since S is compact, the set $C(x)$ is compact. By hypothesis, every three of fewer members of the collection of compact convex sets

$$\{C(x), x \in Q_1\}$$

have at least one point in common. Helly's Theorem [1] then implies that

$$M \equiv \bigcap_{x \in Q_1} C(x) \neq 0. \tag{2}$$

Let $p \in M$. We will prove that S is starshaped relative to p. Firstly, if $p \notin S$, then since S is compact and connected there exists a cross-cut of the complement of S, say uv, such that

$$u \in S,\ v \in S,\ S \cap (uv \sim u \sim v) = 0,\ (p, u, v) \text{ are collinear.} \tag{3}$$

Secondly, if $p \in S$ and if there exists a point $y \in S$ such that

(4) $$py \not\subset S$$

there again exists a cross-cut of the complement of S, uv, such that (3) holds. Hence, to prove $py \subset S$ for each $y \in S$, we prove that (3) cannot hold. Suppose (3) does hold. Since S is closed and connected, the segment uv divides a component of the complement of S, say K, such that $K \sim uv$ contains at least one bounded component. Let K_1 be such a bounded component of $K \sim uv$. This component K_1 of $K \sim uv$ abuts uv from one side of uv. Let H be that closed half-plane whose boundary contains uv and in which K_1 abuts uv. Consider the set

$$K^* \equiv \text{cl conv}(H \cap K_1).$$

Since K^* is a compact convex set with int $K^* \neq 0$, and since int $K^* \subset H$, there must exist a point

$$x \in bd\ K^*$$

and a line of support L to K^* such that

(5) $$L \cap K^* = x, \quad x \in S$$

and such that p and uv lie in the same open half-space bounded by L. Since $u \in S$, $v \in S$, $u \neq v$, since $S = cl$ int S, and since int S is open and connected, for each $\varepsilon > 0$ there exists points $u_1 \in$ int S, $v_1 \in$ int S such that

$$\| u_1 - u \| < \varepsilon, \ \| v_1 - v \| < \varepsilon$$

and such that a polygonal path $P(u_1, v_1)$ exists in int S joining u_1 and v_1. Now, clearly x is a point of local nonconvexity of S otherwise the convexity of $S \cap N$ for some neighborhood N of x would violate condition (5). Furthermore, one can choose a bounded component K_1 of $K \sim uv$ and an $\varepsilon > 0$ (above) so that $K(x)$ must be in the closed half space bounded by L which contains p and uv, because

$$K(x) \cap P(u_1, v_1) = 0.$$

However, in this case, no ray $R(x)$ in conv $K(x)$ exists whose complementary ray $R'(x)$ contains p. This, however contradicts (2), since $p \in M$. Hence, we have arrived at a contradiction. Therefore, S is starshaped.

Since the converse situation is trivial, we have proved Theorem 1.

Theorem 1 has an interesting form when the boundary of S is a simple closed polygon in E_2. To state it we first note the following.

DEFINITION 6. A vertex x of a simple closed polygon P is called reentrant if x is a point of local nonconvexity of the closed bounded polygonal set whose boundary is P.

THEOREM 2. *Let S be a bounded closed set in E_2 whose boundary is a simple closed polygon. Suppose that for each three reentrant vertices of $bd\,S, x_1, x_2, x_3$*

there exist three external rays of support at x_1, x_2, x_3 *respectively to* int S *whose corresponding complementary rays are concurrent* (see (1)) *and meet in* S.

Then S *is starshaped.*

Proof. Theorem 2 follows from Theorem 1 since conditions (a) and (b) are automatically both satisfied.

BIBLIOGRAPHY

1. E. Helly, *Über Mengen konvexer Körper mit gemeinshaftlichen Punkten*, Jber. Deutsch. Math. Verein **32** (1923), 175–176.
2. M.A. Krasnoselskii, *Sur un critère pour qu'un domaine soit étoile*, Math. Sb. (61) 19 (1946)
3. F. Leja and W. Wilkosz, *Sur une propriété des domaines concaves*, Ann. Soc. Polon. Math. **2** (1924), 222–224.
4. J. Molnar, *Über Sternpolygone*. Publ. Math. Debrecen. **5** (1958), 241–245.
5. E.E. Robkin. *Characterizations of starshaped sets*, Doct. Thesis, Univ. of Calif. (1965).
6. F. A. Valentine. *Convex Sets*, McGraw-Hill Book Co. (1964).
7. F. A. Valentine. *Local convexity and* L_n *sets*. (To appear in Proc. Amer. Math. Soc.)

UNIVERSITY OF CALIFORNIA,
LOS ANGELES, CALIFORNIA